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Static and dynamic aspects of disorder lines

T Garel, J C Niel, H Orland and M Schick†

Service de Physique Théorique‡, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

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Abstract. Using one-dimensional models, we characterize the existence of a disorder line (point) in a renormalization group procedure. This procedure is defined in terms of the eigenvalues of the transfer matrix. There are various possibilities for the crossing of subdominant eigenvalues; this study therefore encompasses usual disorder lines (where the spin-spin correlation function changes), as well as other types of disorder lines (where, for instance, the energy-energy correlation function changes). In the usual case, the disorder line shows up as a separatrix between two regions of the flow diagram attracted to different paramagnetic sinks. For this case, we also consider, in the framework of the random phase approximation, the dynamical signature of a disorder line: we show that it can be experimentally detected, especially for disorder lines (points) of the second kind.

1. Introduction

There has been much interest in the concept of disorder lines since they were first introduced by Stephenson [1] and by Fisher and Widom [2] in 1969 and much is known about them [3, and references therein]. For example, the form of the order parameter-order parameter correlation function changes at such a line, typically from a monotonic to a non-monotonic decay, and the correlation length is minimum and non-analytic there [4, 5]. However, all bulk thermodynamic functions are analytic at the disorder line [6]§.

For magnetic systems, to which we restrict our study, the order parameter may be of dipolar nature (e.g. the magnetization) or more complicated (quadrupolar, ...). Following Stephenson [4], one distinguishes disorder lines (or points) of the first (resp. second) kind depending on the finite (resp. zero) value of the relevant correlation length at the disorder condition.

Most of the traditional examples, as reviewed in [3], are concerned with exact solutions in one and two dimensions. A random phase approximation (RPA) approach to usual disorder lines, where the spin-spin correlation function changes, is found in [7]. In order to get a more complete approach to this problem, it is therefore of interest to study how the presence of a disorder line affects a renormalization group (RG) procedure. In a previous paper [8], we have applied the approximate Migdal-Kadanoff scheme to the two-dimensional XY model (for other RG approaches to disorder lines, see [9, 10]), and we have argued that a disorder line would correspond to a change of behaviour of the RG procedure (e.g. from a non-monotonous to a monotonous flow). We focus here our attention on simple one-dimensional Ising models. In section 2, we

† Also at: Department of Physics FM-15, University of Washington, Seattle, WA 98195, USA.

‡ Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique.

§ There can be non-analyticities in the *surface* free energy, however.

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define the RG in terms of the eigenvalues of the transfer matrix. The flow diagram is obtained, and we identify various disorder lines (usual and generalized). These lines correspond to various crossings of the subdominant eigenvalues. Again motivated by [8], we consider in section 3 the possible influence of a usual disorder line on the dynamics of the system. Since we are primarily interested in the paramagnetic phase(s), we use, as in [7], the RPA, and show that disorder lines may indeed influence the dynamics, especially those of the second kind.

2. Renormalization of a one dimensional Ising model

Following [8] and [11], we consider the Ising model shown in figure 1. It contains nearest-neighbour interactions only, with the horizontal interaction between spins in the upper row, $L_0>0$, differing from that between spins in the lower row, $-L_3<0$. These two interactions, one ferromagnetic and the other antiferromagnetic, compete to shape the form of the short-range order.

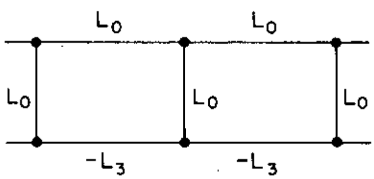

Figure 1. The one-dimensional Ising example studied in section 2.

A renormalization group is defined in terms of the eigenvalues of the transfer matrix T_B of the model. Setting $t=\exp(-L_3/T)$ and $x=\exp(L_0/T)$, these eigenvalues read

$$\lambda_0=b_1+\sqrt{\Delta_1} \qquad \lambda_1=b_1-\sqrt{\Delta_1} \qquad \lambda_2=-b_2+\sqrt{\Delta_2} \qquad \lambda_3=-b_2-\sqrt{\Delta_2}$$

with

$$\Delta_1=b_1^2-\frac{(1-t^4)(1-x^4)}{t^2x^2} \qquad \Delta_2=b_2^2-\frac{(1-t^4)(1-x^4)}{t^2x^2}$$

and

$$b_1=\frac{(1+t^2x^2)(1+x^2)}{2tx^2} \qquad b_2=\frac{(1-t^2x^2)(1+x^2)}{2tx^2}.$$

As explained in detail in [8], they satisfy the following inequalities:

$$\lambda_3<\lambda_1<0<\lambda_2<\lambda_0.$$

As usual, the largest (in modulus) eigenvalue is positive and can thus be identified with λ_0. It corresponds to the free energy, whereas the spin-spin correlation length ξ is given by:

$$\xi^{-1}=\ln\frac{\lambda_0}{|\lambda_s|} \tag{2.1}$$

where $|\lambda_s|$ is the second largest eigenvalue. The sign of λ_s determines the nature of the short range order, i.e. ferromagnetic for $\lambda_s>0$ or antiferromagnetic for $\lambda_s<0$. In order to preserve effective antiferromagnetic couplings, one decimates with a decimation factor $b=3$ to obtain four decoupled recursion relations which can be taken to be

$$G\lambda_0'=\lambda_0^3 \tag{2.2}$$

$$\frac{\lambda_1'}{\lambda_0'}=\left(\frac{\lambda_1}{\lambda_0}\right)^3 \tag{2.3}$$

$$\frac{\lambda_2'}{\lambda_3'}=\left(\frac{\lambda_2}{\lambda_3}\right)^3 \tag{2.4}$$

$$\frac{\lambda_1'}{\lambda_3'}=\left(\frac{\lambda_1}{\lambda_3}\right)^3. \tag{2.5}$$

The first equation is just the usual recursion relation for the free energy per spin, with G a function of the couplings. The second recursion relation has a fixed point at zero as λ_0 is the largest eigenvalue. This leaves two recursion relations of the form $x'=x^3$ with stable fixed points at 0 and $\pm\infty$, and unstable fixed points at ± 1.

A portion of the renormalization group flow is shown in figure 2. Although the system is always paramagnetic, there are two different paramagnetic sinks denoted S_1 and S_2. Systems attracted to the former are characterized by short-range ferromagnetic correlations ($\lambda_s=\lambda_2$), while those attracted to the latter display short-range antiferromagnetic correlations ($\lambda_s=\lambda_3$). These two flows are divided by the disorder line DA, whose equation reads: $\lambda_2=-\lambda_3$. That the disorder line is an invariant subspace of the renormalization group transformation is immediately seen from the recursion relations above (i.e. if $\lambda_2/\lambda_3=-1$, then $\lambda_2'/\lambda_3'=-1$). The existence of this disorder line is guaranteed by the symmetry of the partition function $Z(L_0, L_3)=Z(-L_0, -L_3)$. Indeed the transformation $(L_0, L_3)\to(-L_0, -L_3)$ corresponds to $(x, t)\to(1/x, 1/t)$, which preserves λ_0 and λ_1 and interchanges λ_2 and λ_3. The point D is a singly unstable fixed point which characterizes this separatrix in the flow much like the separatrix at zero magnetic field in the ferromagnetic Ising model. There are, in fact, several examples of the existence of distinct paramagnetic sinks in systems with many interactions [12]. Although separatrices divide the flows and are therefore controlled by fixed points

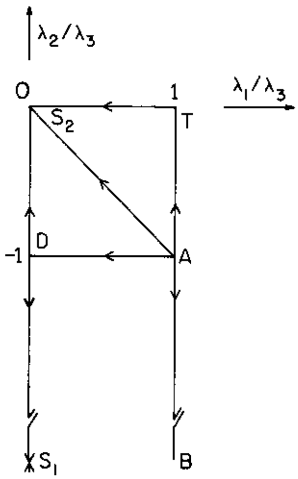

Figure 2. Portion of the renormalization group flow applicable to the model with positive values of L_0 and L_3.

with at least one relevant eigenvalue, there are no singularities in the free energy due to the vanishing of the amplitudes associated with the relevant fields.

One may also consider the crossing of other eigenvalues. For instance, on the line AS_2, the magnitude of λ_1 becomes equal to λ_2, so that one has $\lambda_0 > |\lambda_3| > \lambda_2 = |\lambda_1|$. Below this line the energy–energy correlations are ferromagnetic in character, while above it they are antiferromagnetic. Such a behaviour is found in other models [13, 14]. The line AT is also interesting in that the second largest eigenvalue changes from λ_3, which is negative, to λ_1, which is also negative. That is, the order remains antiferromagnetic. This unusual disorder line cannot be crossed given the parameters of the original model, but can be in the generalized version considered in [8] and [11], a version which contains additional two-spin interactions across the diagonals of the basic rectangle and a four-spin interaction around it. We expect this behaviour to occur whenever there are several interactions which promote the same kind of order and have relative strengths that can be varied. Along the line AB, this same kind of behaviour is found in the energy–energy correlation function.

3. Dynamical effects at disorder lines

3.1. Introduction

Following Stephenson [4], we shall study the one-dimensional Ising examples shown in figure 3. We take in both cases $J_1 > 0$, $J_2 < 0$; moreover, we restrict the range of J_2 (see below), so that the low temperature paramagnetic phase has ferromagnetic short range order. Since we are primarily interested in the paramagnetic phase(s), we use the random phase approximation (RPA). Defining a wavevector dependent susceptibility $\chi(k)$ through

$$\chi(k) = \frac{1}{2\pi} \int G(x)\, e^{-ikx}\, dx \tag{3.1}$$

where $G(x)$ is the spin–spin correlation function $\langle S(0)S(x)\rangle$, one gets the following results [7].

(a) The ANNNI chain (disorder point of the first kind)

$$\chi_a(k) = \frac{1}{T - T_D^{(a)} + 4|J_2|\left(\cos ka - \frac{J_1}{4|J_2|}\right)^2} \tag{3.2}$$

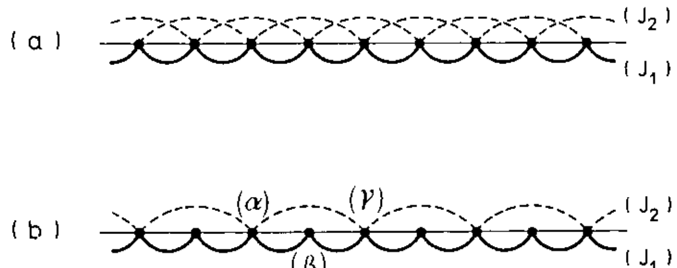

Figure 3. The one-dimensional Ising examples studied in section 3. (*a*) The ANNNI chain (nearest-neighbour and next-nearest-neighbour interactions). (*b*) The decorated chain with alternate interactions (every other spin has next-nearest-neighbour interactions).

where

$$T_D^{(a)} = 2|J_2| + \frac{J_1^2}{4|J_2|} \tag{3.3}$$

is the disorder temperature, and a is the lattice spacing.

(b) The decorated chain, with alternate interactions (disorder point of the second kind).

Decimating spin (β) in figure 3(b), we get an effective interaction between spins (α) and (γ). With the same notation, close to the disorder temperature $T_D^{(b)}$, we have

$$\chi_b(k) = \frac{1}{T_D^{(b)} - (T_D^{(b)} - T)B \cos 2ka} \tag{3.4}$$

where $T_D^{(b)}$ is defined by

$$2J_2 + T_D^{(b)} \ln\left(2 \cosh \frac{2J_1}{T_D^{(b)}}\right) = 0 \tag{3.5a}$$

and B is given by

$$B = 2\alpha \tanh 2\alpha - \ln \cosh 2\alpha \tag{3.5b}$$

with $\alpha = J_1/T_D^{(b)}$.

In the RPA approximation, the above-mentioned restrictions on J_2 read $|J_2| < \lambda J_1$, with $\lambda = \frac{1}{4}$ (case (a)) and $\lambda = 1$ (case (b)).

3.2. The ANNNI chain

3.2.1. Statics. We first consider (3.2), which we rewrite, in the continuum limit, as

$$\chi_a(k) = \frac{1}{\tau_a + |J_2|a^4(k^2 + k_0^2)^2} \tag{3.6}$$

with $\tau_a = T - T_D^{(a)}$ and $(k_0 a)^2 = 2((J_1/4|J_2|) - 1)$. The asymptotic behaviour of the spin-spin correlation function $G(x)$ (3.1) can be obtained by a simple rescaling of the variables. Setting $X = x/a|J_2|^{-1/4}$, $K = ka|J_2|^{1/4}$, $K_0 = k_0 a|J_2|^{1/4}$, we get the following behaviour at large distances ($X\sqrt{|\tau_a|}/2K_0 \gg 1$) and close to $T_D^{(a)}$ ($|\tau_a|$ small):

(i) $\tau_a > 0$
$$G_+(X) \simeq \frac{\pi}{2K_0^3} e^{-K_0 X} \cos\left(\frac{\sqrt{\tau_a} X}{2K_0}\right) \tag{3.7a}$$

(ii) $\tau_a < 0$
$$G_-(X) = \frac{\pi}{2K_0^3} e^{-K_0 X} \operatorname{ch}\left(\frac{\sqrt{|\tau_a|} X}{2K_0}\right) \tag{3.7b}$$

(iii) $\tau_a = 0$
$$G_D(X) = \frac{\pi}{2K_0^3} X e^{-K_0 X}. \tag{3.7c}$$

These expressions are valid, provided that

$$\frac{\sqrt{|\tau_a|}}{K_0} \ll K_0 \tag{3.8a}$$

and

$$\frac{X\sqrt{|\tau_a|}}{K_0} \gg 1. \tag{3.8b}$$

Condition (3.8a) implies that $T_C^{(a)}$ (in the RPA scheme) and $T_D^{(a)}$ are well separated since $T_D^{(a)} - T_C^{(a)} = +K_0^4$. Condition (3.8$b$) implies that

$$\frac{x}{a} \gg \frac{1}{k_0 a} \sqrt{\frac{T_D^{(a)} - T_C^{(a)}}{|T - T_D^{(a)}|}}. \tag{3.9}$$

For $T_D^{(a)} = 2T_C^{(a)}$, we get

$$\frac{x}{a} \gg \frac{1}{k_0 a\sqrt{2}} \sqrt{\frac{T_D^{(a)}}{|T - T_D^{(a)}|}}$$

which is not too stringent a condition for, e.g., $T = \frac{3}{4}T_D^{(a)}$ and $T = \frac{5}{4}T_D^{(a)}$.

3.2.2. Dynamics. Having recalled some known facts about $T_D^{(a)}$, we now consider the following Langevin equation for the time-dependent k mode of the magnetization S:

$$\frac{\partial S_k(t)}{\partial t} = -\Gamma_0(\chi_a(k))^{-1} S_k + \eta_k(t) \tag{3.10}$$

with $\langle \eta_k(t)\eta_{k'}(t')\rangle = 2\Gamma_0 T\delta_{kk'}\delta(t-t')$. Using (3.10), we obtain the (X, ω) spin–spin correlation function

$$\mathscr{S}(X,\omega) = \langle S(X,\omega)S(0,0)\rangle \simeq \int \frac{e^{iKX}}{\omega^2 + [\tau_a + (K^2 + K_0^2)^2]^2}\,\mathrm{d}K \tag{3.11}$$

where X and K are given above. With the use of equations (3.7), we now get for the $\omega = 0$ value of $\mathscr{S}(X,\omega)$ at large X and close to $T_D^{(a)}$ (cf (3.9)),

$$\text{(i)} \qquad \tau_a > 0 \qquad \mathscr{S}(X,\omega=0) \simeq \frac{\pi}{32K_0^5} X\,e^{-K_0 X} \frac{\sin\left(\frac{\sqrt{\tau_a}}{2K_0}X\right)}{\left(\frac{\sqrt{\tau_a}}{2K_0}\right)} \tag{3.12a}$$

$$\text{(ii)} \qquad \tau_a < 0 \qquad \mathscr{S}(X,\omega=0) \simeq \frac{\pi}{32K_0^4} X\,e^{-X/\xi_-} \frac{1}{\sqrt{|\tau_a|}} \tag{3.12b}$$

where ξ_- is the correlation length below $T_D^{(a)}$, and

$$\text{(iii)} \qquad \tau_a = 0 \qquad \mathscr{S}(X,\omega=0) \simeq X^3\,e^{-K_0 X}. \tag{3.12c}$$

The disorder point ($\tau_a = 0$) is thus characterized by an asymmetric behaviour of the correlation function $\mathscr{S}(X, \omega = 0)$, since, for large X, $\sin ux/u$ behaves like $\delta(u)$ [see (3.12a)]. The 'large X' behaviour is given in (3.8b) and the vicinity of $T_D^{(a)}$ in (3.8a). We note that this asymmetry in a dynamic correlation function is not unexpected since the slope of the static correlation length ξ is infinite for $\tau_a \to O^-(\xi_-)$ and finite for $\tau_a \to O^+(\xi_+)$. The diffusion coefficient should therefore be asymmetric with respect to $\tau_a \to 0$. A similar result would be obtained by using the exact (static) solution [4].

3.3. The decorated chain with alternate interactions

More direct calculations are possible in this case, due to the simple form of (3.4). In particular since $\xi = 0$ at $T_D^{(b)}$, with a vertical slope on both sides, we obtain a vanishing diffusion coefficient at the disorder temperature. In this case, it is also possible to characterize $T_D^{(b)}$ as a point where the distance between two spin configurations submitted to the same thermal noise [15] is slower than in the rest of the paramagnetic phase. Let us consider two such spin configurations, which we label by indexes 1 and 2. Equation (3.10) reads

$$\frac{\partial S_k^{(1)}}{\partial t} = -\Gamma_0(\chi_b(k))^{-1} S_k^{(1)} + \eta_k(t) \tag{3.13a}$$

$$\frac{\partial S_k^{(2)}}{\partial t} = -\Gamma_0(\chi_b(k))^{-1} S_k^{(2)} + \eta_k(t). \tag{3.13b}$$

Defining $\mu_k = S_k^{(1)} - S_k^{(2)}$, we have

$$\mu_k(t) = \mu_k(0)\,\mathrm{e}^{-\Gamma_0(\chi_b(k))^{-1}t}. \tag{3.14}$$

The distance between configurations (1) and (2) is defined as

$$d(t) = \frac{1}{L}\sum_x \{S^{(1)}(x,t) - S^{(2)}(x,t)\}^2 \tag{3.15a}$$

$$= \frac{1}{L}\sum_x \mu^2(x,t) \tag{3.15b}$$

with $\mu(x,t) = 1/L\,\Sigma_k\,\mu_k(t)\,\mathrm{e}^{ikx}$. Here L is the size of the system (due to our convention as given in figure 3(b), we take the original lattice spacing a to be one half) and k belongs to the first Brillouin zone $[-\pi, \pi]$. Note that the above definition yields $d(t) \sim O(1/L)$ if configurations (1) and (2) differ by a finite number of spins. We therefore get

$$d(t) = \frac{1}{L^2}\sum_k \mu_k(0)\mu_{-k}(0)\,\mathrm{e}^{-2\Gamma_0 t(\chi_b(k))^{-1}}. \tag{3.16}$$

At $T = T_D^{(b)}$, we have $(\chi_b(k))^{-1} = T_D^{(b)}$. The distance $d(t)$ then reads

$$d(t) = d(0)\,\mathrm{e}^{-2\Omega_0 t} = \mathrm{e}^{-2\Omega_0 t}\,\frac{1}{L}\sum_x \mu^2(x,0) \tag{3.17}$$

with $\Omega_0 = \Gamma_0 T_0^{(b)}$. For T smaller (resp. larger) than $T_D^{(b)}$, the effective interaction between spins (α) and (γ) of figure 3(b) is ferromagnetic (resp. antiferromagnetic). We therefore expand, to lowest order, the k dependent part of $\chi_b(k)$ (see (3.4)) around $k=0$ (resp. $k=\pi$). We get

$$d(t) = \frac{\mathrm{e}^{-2\Omega_0 t}}{L^2}\sum_k \sum_{x,x'} \mu(x,0)\mu(x',0)\,\mathrm{e}^{ik(x-x')}\,\mathrm{e}^{-D_b k^2 t} \tag{3.18}$$

with a diffusion coefficient $D_b = 4\Gamma_0 B(T_D^{(b)} - T)$ (resp. $D_b = 4\Gamma_0 B(T - T_D^{(b)})$). In the continuum limit, the discrete sum over k becomes

$$\frac{L}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}k\,\mathrm{e}^{ik(x-x')}\,\mathrm{e}^{-D_b k^2 t}$$

which, for t large ($t \gg (D_b)^{-1}$), is well approximated by extending the limits of the integral to infinity. We thus obtain

$$d(t) \simeq \frac{1}{L}\frac{1}{\sqrt{D_b t}}\,\mathrm{e}^{-2\Omega_0 t}\sum_{x,x'} \mu(x,0)\mu(x',0)\exp\left(-\frac{(x-x')^2}{4D_b t}\right) \tag{3.19}$$

for t large.

Choosing for instance $\mu(x,0)=S^{(1)}(x,0)-S^{(2)}(x,0)=\delta_{x,0}$ (the configurations differ by a single spin), we find for large times

(i) $T=T_D^{(b)}(D_b=0)$

$$d(t)\simeq\frac{1}{L}\,e^{-2\Omega_0 t} \qquad (3.20)$$

(ii) $T\neq T_D^{(b)}(D_b\neq 0)$

$$d(t)\simeq\frac{1}{L}\frac{e^{-2\Omega_0 t}}{\sqrt{D_b t}} \qquad (3.21)$$

which shows how the dynamics is 'slowed' down close to a disorder point of the second kind.

Thus by using the RPA, which should be quite accurate in the high temperature phase, we have exhibited one dimensional models where some dynamics is sensitive to the occurrence of a disorder point.

4. Conclusion

We have considered some aspects of disorder lines, both from the RG and RPA points of view, in simple one-dimensional models. We have shown in particular that they may have interesting effects on the dynamics. This certainly can be extended to higher dimensional models where disorder lines play an important role [16].

Acknowledgments

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References

[1] Stephenson J 1969 *Can. J. Phys.* **47** 2621
[2] Fisher M E and Widom B 1969 *J. Chem. Phys.* **50** 3756
[3] Batchelor M T and van Leeuwen J M J 1989 *Physica* **154** 365
[4] Stephenson J 1970 *Can. J. Phys.* **48** 1724
[5] Stephenson J 1970 *J. Math. Phys.* **11** 420
[6] Gompper G and Schick M 1990 (unpublished)
[7] Garel T and Maillard J M 1986 *J. Phys. C: Solid State Phys.* **19** L505
[8] Garel T, Niel J C and Orland H 1990 *Europhys. Lett.* **11** 349
[9] Azaria P, Diep H T and Giacomini H 1989 *Europhys. Lett.* **9** 755
[10] Le Doussal P and Harris A B 1988 *Phys. Rev. Lett.* **61** 625
[11] Hornreich R M, Liebmann R, Schuster H G and Selke W 1979 *Z. Phys.* B **35** 91
[12] Berker A N and Wortis M 1976 *Phys. Rev.* **14** 4946
[13] Thorpe M F and Blume M 1972 *Phys. Rev.* B **5** 1961
[14] Wang Y-L, Lee F and Kimel J D 1987 *Phys. Rev.* B **36** 8945
[15] Derrida B and Weisbuch G 1986 *J. Physique* **47** 1297
[16] Gompper G and Schick M 1989 *Phys. Rev. Lett.* **62** 1647; 1990 *Phys. Rev.* B **41** 9148